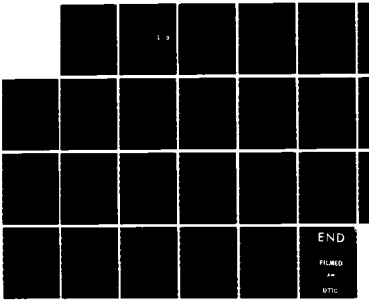
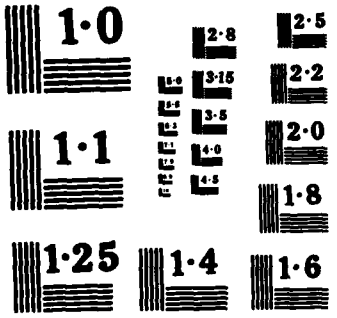
AD-A162 933 ON DOMAIN DECOMPOSITION(U) STANFORD UNIV CA CENTER FOR LARGE SCALE SCIENTIFIC COMPUTATION C R ANDERSON OCT 85 CLASSIC-85-09 N00014-82-K-0335 1/1

UNCLASSIFIED F/G 12/1 NL





NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

(15)

CLaSSiC Project
Manuscript CLaSSiC-85-09

October 1985

On Domain Decomposition

Christopher R. Anderson

Center for Large Scale Scientific Computation
Building 460, Room 313
Stanford University
Stanford, California 94305

On Domain Decomposition

Christopher R. Anderson

Department of Mathematics

and

Department of Computer Science

Stanford University

Stanford, California 94305

Partially supported by NSF Mathematical Sciences Postdoctoral Research Fellowship #MCS 83-11685 and ONR contract N00014-82-K-0335.

In this report we discuss aspects of the method of domain decomposition for the solution of elliptic boundary value problems in two dimensions. The basic idea behind the technique is to piece together local solutions of the elliptic problem to form a global solution. There are several reasons for wanting to use such a procedure. The first is that a given domain may be irregular, but can be subdivided into regular pieces for which solutions are computationally efficient to obtain. Another reason is that the method may be suitable for for solving elliptic problems on multiple processor machines. If, for example, one breaks up the domain into many small pieces and allocates a processor to each piece, then there is the possibility of decreasing the computational time by constructing the local solutions in parallel and "gluing" them together. The topic of domain decomposition has become increasingly popular recently and many papers have appeared on the subject. The specific domain decomposition method that we shall be concerned with is that discussed and analysed by Dryja [8], Bjorstad and Widlund [1], Golub and Mayers [9], and Bramble, Pasciak and Hubbard [2]. The paper by Bjorstad and Widlund has an extensive bibliography as well as a short summary of the history of the method. The purpose of this report is to examine model problems with the intention of providing motivation and insight concerning results previously presented as well as offer some new observations that may aid in the implementation of the method.

Earlier papers primarily view domain decomposition from a discrete viewpoint - the method is applied to the equations resulting from a discretization of the elliptic problem. However, underlying the method for the discrete equations is a method for the continuous equations. We shall examine the method for the continuous equations. We are interested in this viewpoint because it provides physical interpretations for the operators that arise from the discrete application. Also, conclusions derived for the continuous equation should be inherited by any consistent discretization. For problems with relatively simple geometry (i.e. the problems we are considering here) explicit representations of the operators occurring in the method can be found. This aids in the understanding of the behavior of the method, especially with respect to changes in the geometry of the domain.

In the discrete version of domain decomposition, one is faced with the problem of finding the solution to a particular set of linear equations. (The equations described by the capacitance matrix.) For reasons which will be discussed, it is advantageous to solve the system using an iterative technique. The iterative technique which appears to be the most popular is the method of pre-conditioned conjugate gradients [4]. The success of this iterative scheme relies on the choice of a pre-conditioner. This is an operator whose inverse is an approximate inverse for the linear system. It is our aim to provide some insight into the nature of several pre-conditioners that have been proposed, and the effect that certain parameters in the elliptic problem have on the rate of convergence of the resulting iterative scheme. Also, perhaps more importantly, we shall give some insight into the physical significance of these preconditioners and demonstrate that, although they may be defined in different forms, they are all essentially the same.

The continuous approach that we follow uses integral equations. We also work with the differential form of the equations. Another continuous approach, and that used in [1] and [2], is the variational formulation of the elliptic problem. We use the integral equation approach because in the case of simple geometry we are able to find explicit representations of the operators involved. We also found it more useful in providing motivation for the method. (In particular, it is much easier to interpret various preconditioners using such an analysis.) The variational formulation, as demonstrated in [1] and [2] is more general, being able to deal with variable coefficient elliptic problems, and perhaps more useful for proving certain properties associated with method. For example, Bjorstad and Widlund [1] prove the convergence of the conjugate gradient algorithm for the capacitance matrix equations when their suggested preconditioner is used. It is possible that one could use the integral equation approach given here to prove results concerning domain decomposition, however, it seems that the variational formulation is better suited to such a task.

In the first section we give a brief description of the method of domain decomposition from the discrete viewpoint. We then describe the method from a continuous viewpoint. Using the continuous approach we examine some suggested preconditioners. The continuous approach is then used to examine the extension of the domain decomposition ideas for a variable coefficient elliptic problem. In the fifth section we give a derivation of the discrete version of some of the results obtained for the continuous case. This discrete version is based on an underlying 5-point difference scheme. Pieces of this derivation first appeared in [1] and more recently a complete derivation has appeared in Chan [6] and Chan and Resasco [7]. We present our derivation for the convenience of the reader. It is also slightly different than those previously presented. The results of the derivation are of value to those doing domain decomposition on rectangles - the results provide the information necessary for an implementation using fast Fourier transforms.

Acknowledgements The author wishes to thank Gene Golub and Walter Craig for the many conversations concerning domain decomposition that he has had with them. The author also wishes to thank Claude Greengard for valuable comments concerning an early draft of this report.

1. Description of the Method

In this Section we present the basic ideas of domain decomposition. We first work with the discrete version of the method and then the continuous version.

The problem upon which we shall focus primarily is the solution of Poisson's equation on a domain consisting of two abutting rectangular regions. Ω denotes the domain and Ω_1 and Ω_2 the subdomains. Γ is the interface between the two subdomains. (See Figure 1.) The equation to be solved is

$$\Delta u = f$$
$$u = g \quad \text{on } \partial\Omega \tag{1.1}$$

The first step in the method of domain decomposition is to calculate the values of the solution along Γ. The two subproblems

$$\Delta u_i = f \quad \text{in } \Omega_i \tag{1.2}$$
$$u_i = u \quad \text{on } \Gamma \quad \text{and } u_i = g \quad \text{on } \partial\Omega_i \,/\, \Gamma$$

are then solved. The solution on Ω is the union of the two solutions u_i.

To implement this idea it is necessary to find and solve an equation for u along Γ. We shall call this equation the interface equation. It may seem unlikely that such an equation will exist. What we shall find, at least for the model problem (and it can be expected to be true for other problems), is that there is an equation for u along Γ. The form of this equation depends on the geometry of the domain while the influence of the data, f and g, and of the values of u in each of the subregions will enter into the equation as forcing functions.

One method of deriving the interface equation is to work with a discretized version of problem (1.1). This approach is presented by Dryja [8], Golub and Mayers [9], and Bjorstad and Widlund [1]. Consider a standard five-point discretization of (1.1) written in the form

$$\frac{1}{h}\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^t & A_{23}^t & A_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{1.3}$$

Here we have cancelled one factor of $\frac{1}{h}$ from both sides of the equation. The values of u are lumped into three groups X_1, X_2 and X_3 corresponding to points in Ω_1, Ω_2, and along Γ respectively. A_{11} contains the coefficients of the equations for the solution values at points in Ω_1, A_{22} the coefficients for the values at points in Ω_2 , and A_{33} contains the coefficients for the values at points along Γ. The coefficients representing the coupling between the different sets of points is contained in A_{13}, A_{23} and the transpose of these matrices. One derives an equation for the values X_3 by doing block Gaussian elimination on the matrix system (1.3). (This is just Gaussian elimination on (1.3) thinking of the sub-matrices as numbers.) The resulting equation is

$$\left[\ C\ \right]\left[\ X_3\ \right] = \frac{1}{h}\left[\ A_{33} - A_{13}^t A_{11}^{-1} A_{13} - A_{23}^t A_{22}^{-1} A_{23}\ \right]\left[\ X_3\ \right] = F_3 - A_{13}^t A_{11}^{-1} F_1 - A_{23}^t A_{22}^{-1} F_2 \tag{1.4}$$

This system of equations, represented by the matrix C, is the Shur compliment of A_{33} with respect to A_{11} and A_{22}. C is often called the capacitance matrix. It can be shown that this system is negative definite if the original system is. [5] Equation (1.4) is the interface equation that we are seeking. It is the solution of this system which is the principle difficulty encountered in the method of domain decomposition. Once a solution is found, the solution values on either side of Γ can be obtained by solving the equations corresponding to each of the sub-domains and using X_3 as data along Γ. Unfortunately, the matrix C is difficult to construct explicitly (in general). We note that the matrix involves A_{11}^{-1} and A_{22}^{-1}, which are usually only known implicitly - they are often subroutine calls. Of course one could construct C by applying C to a set of basis vectors along Γ. Assuming N points in Ω and approximately $N^{1/2}$ points along Γ, this would involve approximately $N^{1/2}$ applications of A_{11}^{-1} and A_{22}^{-1}.

If the inverses of A_{11} and A_{22} were, in the best possible case, implemented using fast solvers, then the resulting work would be the order of $N^{3/2}LogN$. This may be acceptable for some problems, but the attempt is made to solve (1.4) using a method that does not require an explicit representation of C. In light of this requirement, iterative methods are a natural choice for the solution of (1.4).

One very popular iterative technique is the generalized, or preconditioned conjugate gradient algorithm [4]. Given the problem of solving $A\,x = b$, one can think of the generalized method as the conjugate gradient scheme applied to the prepared system

$$M^{-1}\,Ax = M^{-1}b$$

in the inner product defined by

$$[x\,,\,y\,] = (x,\, M^{-1}y)$$

where M is the preconditioner and $(\cdot,\,\cdot)$ is the standard L^2 inner product [2]. An error estimate [4] for the resulting iterative procedure is

$$\frac{(x^n - x)^t\ A\,(x^n - x)}{(x^0 - x)^t\ A\,(x^0 - x)} \leq 4\left(\frac{\kappa^{\frac{1}{2}} - 1}{\kappa^{\frac{1}{2}} + 1}\right)^{2n} \tag{1.5}$$

κ is the condition number; the ratio of the largest eigenvalue to the smallest eigenvalue of $M^{-1}A$

$$\kappa = \frac{\lambda_{max}(M^{-1}A)}{\lambda_{min}(M^{-1}A)}\ .$$

From this error estimate we infer that a good preconditioner is one which is a multiple of an approximate inverse of A.

The success of the pre-conditioned conjugate gradient method is dependent upon the choice of the preconditioner. For the model problem of two abutting rectangular regions, there have been several pre-conditioners suggested for the matrix equation (1.4). We shall be interested in three of them. The first two assume a uniform rectangular grid is used and that

the nodes on the interface are an equal distance apart.

Dryja's and Golub and Mayers' preconditioner are defined in terms of the matrix K, the matrix corresponding to the standard 3-point discretization of the one dimensional Laplacian along Γ.

$$K = \frac{-1}{h^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & \cdot & & \\ & & & \cdot & 1 \\ & & & 1 & -2 \end{pmatrix}$$

The suggested preconditioner is to use

$$M = -(4K)^{\frac{1}{2}}$$

Here $(\ )^{\frac{1}{2}}$ represents the square root of the matrix K. The second preconditioner, one suggested by Golub and Mayers in [9], is a modification of Dryja's. Specifically, they use

$$M = -(4K + h^2K^2)^{\frac{1}{2}}$$

The third, suggested by Bjorstad and Widlund, is to use as M^{-1} the operator T defined implicitly as follows:

Let Neumann data η be specified on Γ. Solve in Ω_1 the problem

$$\Delta\phi = 0 \quad \text{with} \quad \frac{\partial\phi}{\partial n} = \eta \ \text{ on } \ \Gamma$$

$$\phi = 0 \ \text{ on } \ \partial\Omega_1 \ / \ \Gamma$$

Define δ to be ϕ evaluated along Γ. T is the operator representing the mapping

$$T{:}\eta \rightarrow \delta$$

i.e. the mapping that takes Neumann data on Γ to Dirichlet data on Γ. In matrix terms this means solving the equation

$$\begin{pmatrix} A_{11} & A_{13} \\ A_{13}^t & A_{33}^{(1)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$$

and setting $\delta = Y_3$. Here $A_{33}^{(1)}$ is the appropriate boundary discretization for a Neumann problem on Ω_1. There is nothing special about using Ω_1 to define the preconditioner, one could define it using Ω_2 as well.

For convenience, we shall refer to Dryja's preconditioner as D, that due to Golub and Mayers as GM, and the two suggested by Bjorstad and Widlund, one defined using the Neumann - Dirichlet mapping on Ω_1 and the other on Ω_2, as BW1 and BW2 respectively.

There are several questions related to the use of these preconditioners. The first question which comes to mind is "Why are these good preconditioners?" Other questions concern the relative merits of the preconditioners. "Are there substantial differences between them?" "Do they work well for some configurations of two rectangles, but not others?" "How easy are they to implement?" Also, "How well do these preconditioners work as the mesh corresponding to a discretization tends to zero?" And lastly, "For Bjorstad and Widlund's preconditioner, which piece of the domain should be used?"

Some questions have been answered. For example, in the case of the problem considered here, numerical experiments presented in [9] and [1] indicate that each of the above preconditioners is very good. Typically, the conjugate gradient iteration converges in a few iterations. There is no clear best preconditioner. However, it appears that preconditioners GM and BW1 or BW2 are better than D. As for theoretical results, Dryja [8] and Bjorstad and Widlund [1] have been able to obtain bounds on the condition number of $M^{-1}A$ for preconditioners D and BW1. If the domain given in Figure 1 is discretized using a standard five point approximation to the Laplacian, then the bounds derived by Bjorstad and Widlund are

$$\kappa(\mathrm{D}) \leq \frac{3}{\pi}\left(\frac{q+1}{n+1} + \frac{q+1}{r+1}\right) + 2\sqrt{2} \tag{1.6}$$

$$\kappa(BW1) \leq 1 + \sqrt{2} + \frac{3}{\pi}\left(\frac{q+1}{n+1}\right)$$

Here p,q,r,m, and n are the number of interior nodes corresponding to sides of the domain. In

Figure 2 the correspondence is made explicit. Dryja also presents a bound for $\kappa(D)$ which is similar to that given above. One interesting aspect of these bounds is the condition number for both preconditioners becomes unbounded if the ratio $\frac{q+1}{n+1}$ does. Bjorstad and Widlund conjecture that more refined estimates may not have this behavior.

For problems on more general regions, Bjorstad and Widlund [1] prove that for the continuous analogs of preconditioner D and BW1 the condition number of $M^{-1}A$ is bounded. (The continuous analog used is based on the equivalent variational formulation of the problem.) If the bounds (1.6) are sharp, then this result is consistent as long as the refinement of the underlying mesh is carried out in such a way that $\frac{q+1}{n+1}$ and $\frac{q+1}{r+1}$ remain constant. (This is a very natural way to refine.)

As for the simplicity of computation, if one has a problem of two abutting rectangles in which the interface is a straight line, then, for a five point discretization, all of the preconditioners can be implemented using fast sine transforms on Γ. In fact, every step after the first of the conjugate gradient iteration can be carried out using fast sine transforms on the line. For the specific details see [5], [6] or Section 5. When the boundary is curved it is no longer possible to use the fast sine transforms. Bjorstad and Widlund's preconditioner, not depending upon the boundary being a straight line, is then perhaps the easiest to apply. All that is needed to implement their preconditioner is a Neumann solver for one of the domains.

To help answer the remaining questions, i.e. motivation for the preconditioners and related issues, we shall consider the domain decomposition method from a continuous viewpoint. The basic procedure is the same as in the discrete case, except that instead of matrices we shall have integral equations.

For a domain as in Figure 1, the principle idea is to construct a solution which is the composed of two solution u_1 and u_2. These functions are solutions of the two subproblems given by equation (1.2). The boundary data for u_1 and u_2 along Γ is the value of the solution

u along Γ. Over the rest of the boundary, u_1 and u_2 take the boundary values g. We shall derive an equation for u along Γ by considering the conditions that u_1 and u_2 must satisfy in order to form a solution on the whole domain. Clearly we must have u_1 and u_2 equal along Γ. The other condition is the continuity of their normal derivatives. This condition is a necessary and sufficient condition that the two solutions u_1 and u_2 must satisfy in order to form a weak solution of the equations.

By construction u_1 and u_2 are continuous along Γ. What is not guaranteed is that the normal derivatives along the interface will be continuous. The normal derivatives implicitly depend on the values of u along Γ, and the requirement of their continuity will translate into our interface equation. We now use Green's functions to find an explicit representation for this equation.

For some technical reasons, which will become clear later, it is useful for us to consider as our unknown the difference of u along Γ from a linear function which interpolates the boundary values of u at the end points of Γ. Thus, we shall derive an equation for u' where

$$u' = u - u_l$$

We derive our equation by finding expressions for $\frac{\partial u_1}{\partial n}$ and $\frac{\partial u_2}{\partial n}$ along Γ and then setting them equal. We denote points in R^2 as $x = (x_1, x_2)$ or $y = (y_1, y_2)$. We shall consider the normal derivative, $\frac{\partial}{\partial n}$, to be taken using the outward normal with respect to Ω_1. $\frac{\partial}{\partial n_x}$ applied to a function of two sets of variables, say $F(x, y)$, refers to a normal derivative taken with respect to the x variables. To find an expression for $\frac{\partial u_1}{\partial n}$ we utilize Green's identity.

Let G_1 be the Green's function for Ω_1, that is

$$\Delta G_1(x, y) = \delta(x - y)$$

and $G_1(x, y) = 0$ for all y on $\partial\Omega_1$. Here δ is Dirac's delta function. We have the identity

$$u_1(x) = \int_{\Omega_1} G_1(x, y) f_1(y) dy + \int_{\partial\Omega_1} u_1(s) \frac{\partial G_1}{\partial n_y}(x, s) ds \tag{1.7}$$

$$= \int_{\Omega_1} G_1(x, y) f_1(y) dy + \int_{\partial\Omega_1} \tilde{g}_1(s) \frac{\partial G_1}{\partial n_y}(x, s) ds + \int_{\partial\Omega_1} u'(s) \frac{\partial G_1}{\partial n_y}(x, s) ds$$

f_1 is the restriction of f to Ω_1, and we have broken up the boundary values of u_1 on $\partial\Omega_1$ into two pieces, the first piece, $\tilde{g}_1$, defined by

$$\tilde{g}_1 = \begin{cases} g & \text{on } \partial\Omega_1 / \Gamma \\ u_l & \text{on } \Gamma \end{cases}$$

and the second piece equal to u' on Γ. By taking the normal derivative of (1.7) and evaluating it along Γ we have

$$\frac{\partial u_1}{\partial n_x}(x) = \frac{\partial}{\partial n_x} \left[\int_{\Omega_1} G_1(x, y) f_1(y) dy + \int_{\partial\Omega_1} \tilde{g}_1(s) \frac{\partial G_1}{\partial n_y}(x, s) ds \right]$$

$$+ \int_{\Gamma} u'(s) \frac{\partial^2 G_1}{\partial n_x \, \partial n_y}(x, s) ds$$

(We have assumed that it is valid to differentiate under the integral sign.) A similar expression holds for $\frac{\partial u_2}{\partial n_x}$ in Ω_2. Setting these expressions equal and collecting the integrals that contain u' on the left hand side, we obtain an integral equation along Γ for u'

$$\int_{\Gamma} \left[\frac{\partial^2 G_1}{\partial n_x \, \partial n_y} + \frac{\partial^2 G_2}{\partial n_x \, \partial n_y} \right](x, s) u'(s) ds = \tag{1.8}$$

$$\frac{\partial}{\partial n_x} \left[\int_{\Omega_1} G_1(x, y) f_1(y) dy + \int_{\partial\Omega_1} \tilde{g}_1(s) \frac{\partial G_1}{\partial n_y}(x, s) ds \right]$$

$$+ \frac{\partial}{\partial n_x} \left[\int_{\Omega_2} G_2(x, y) f_2(y) dy + \int_{\partial\Omega_2} \tilde{g}_2(s) \frac{\partial G_2}{\partial n_y}(x, s) ds \right]$$

The right hand side of (1.8) has the interpretation of being the jump across Γ of the normal derivative of the solutions to two Dirichlet problems. For the term

$$\frac{\partial}{\partial n_x}\left[\int_{\Omega_1} G_1(x\,,\,y)f_{\,1}(y)dy \;+\; \int_{\partial\Omega_1} \tilde{g}_1(s)\,\frac{\partial G_1}{\partial n_y}\,(x\,,\,s)ds\right] \tag{1.9}$$

is the normal derivative of ϕ, the solution to

$$\Delta\phi = f_{\,1} \qquad \phi = \tilde{g}_1 \;\text{ on } \partial\Omega_1$$

and a similar interpretation holds for Ω_2. The operator on the left hand side of (1.8) is also composed of two operators, one associated with each domain. The operator associated with Ω_i ,

$$\int_{\Gamma} \frac{\partial^2 G_1}{\partial n_x\;\partial n_y}(x\,,\,s)u'\;(s)ds \tag{1.10}$$

takes data u' on Γ to the normal derivative of the solution of

$$\Delta\,\psi_i = 0 \quad \text{in } \Omega_i \tag{1.11}$$

$$\psi_i = u' \quad \text{on } \Gamma \qquad \psi_i = 0 \text{on } \;\Omega_i \;/\,\Gamma$$

Since each operator takes data on Γ to flux values, we shall refer to these operators as 'flux' operators. Thus equation (1.8) expresses the fact that the jump of the normal derivatives (or flux) of u_1 and u_2 induced by boundary values u' along Γ must equal the jump in normal derivative (or flux) induced by the forcing function f_i in each region and the boundary values $\tilde{g}_1$ and $\tilde{g}_2$.

In the case of the five-point difference approximation to the Laplacian, there is a direct correspondence between the matrix problem (1.4) and equation (1.8):

$$\left[A_{33} - A_{13}^t A_{11}^{-1} A_{13} - A_{23}^t A_{22}^{-1} A_{23} \right] =$$

$$= \left[A_{33}^1 + \tfrac{1}{2} D - A_{13}^t A_{11}^{-1} A_{13} \right] + \left[A_{33}^2 + \tfrac{1}{2} D - A_{23}^2 A_{22}^{-1} A_{23} \right]$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_\Gamma \frac{\partial^2 G_1}{\partial n_x \, \partial n_y}(x, s) u'(s) ds + \frac{\partial^2 G_2}{\partial n_x \, \partial n_y}(x, s) u'(s) ds$$

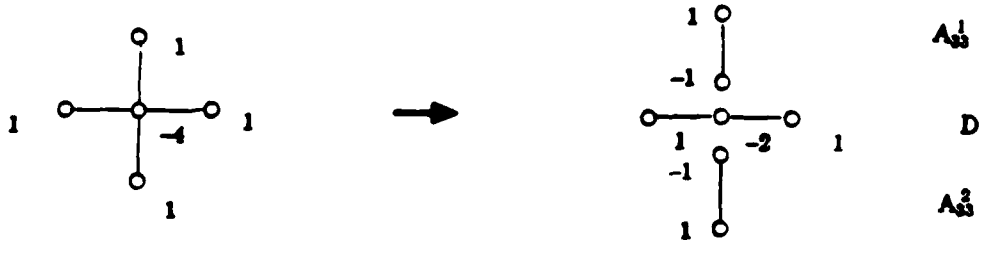
and for the right hand side,

$$\tfrac{1}{2} F_3 - A_{13}^t A_{11}^{-1} F_1 + \tfrac{1}{2} F_3 - A_{23}^t A_{22}^{-1} F_2$$

$$\swarrow \qquad\qquad \searrow$$

$$\frac{\partial}{\partial n_x} \left[\int_{\Omega_1} G_1(x, y) f_1(y) dy + \int_{\partial\Omega_1} \tilde{g}_1(s) \frac{\partial G_1}{\partial n_y}(x, s) ds \right] + \frac{\partial}{\partial n_x} \left[\int_{\Omega_2} G_2(x, y) f_2(y) dy + \int_{\partial\Omega_2} \tilde{g}_2(s) \frac{\partial G_2}{\partial n_y}(x, s) ds \right]$$

Here we have decomposed A_{33} as $A_{33} = A_{33}^1 + D + A_{33}^2$. A_{33}^1 contains the entries of A_{33} that represent the coupling of the nodes of Γ to Ω_1. Similarly A_{33}^2 corresponds to the coupling of nodes of Γ to Ω_2. D contains the entries which correspond to the coupling of nodes of Γ to each other. For example, consider one node on Γ, then the entries of the five point difference stencil associated with the node would be broken up as



2. Explicit Representation of the Equations

In the derivation of (1.8) the fact that the domain was the union of two rectangles was not used. However, it is for such simple regions that the integral equation (1.8) can be put in more explicit form. Before we consider the case of two abutting rectangles, it is instructive to consider the equation for an even simpler geometry, namely that of the half-plane.

The problem to be solved is

$$\Delta u = f \quad \text{in } R^2$$

We consider R^2 broken up into two pieces - the upper and lower half planes. We are primarily interested in finding a representation for the operator on the left hand side of (1.8). The right hand side can be obtained by evaluating the jump in normal derivative of solutions of a Dirichlet problem in each region. (See the discussion concerning equations (1.8) and (1.9).) The Green's function for the upper half plane is given by

$$G(x, y) = \frac{1}{2\pi}\text{Log}([(x_1 - y_1) + (x_2 - y_2)]^{\frac{1}{2}}) - \frac{1}{2\pi}\text{Log}([(x_1 - y_1) + (x_2 + y_2)]^{\frac{1}{2}})$$

The Green's function for the lower half plane is similar. If one follows the steps that led to (1.8) with these Green's functions, then one arrives at an integral equation of the form

$$\int_{-\infty}^{+\infty} u'(\xi)\left(\frac{1}{\pi(s-\xi)^2} + \frac{1}{\pi(s-\xi)^2}\right)d\xi = \frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{u'(\xi)}{(s-\xi)^2}d\xi = R(s) \qquad (2.1)$$

where R(s) represents the jump in normal derivative along the x_1-axis due to the half-space problems

$$\Delta\psi_i = f_i \quad \text{in } \Omega_i$$

$$\psi = 0 \quad \text{on } x_1 = 0$$

for $i = 1,2$. Ω_1 and Ω_2 are the upper and lower half plane respectively.

The interface equation can therefore be represented as a singular integral operator equation. We note that the operator (2.1) is a convolution with a shift invariant kernel, and hence

the Fourier transform of the equation is represented by multiplication of the transform of u' with the transform of the kernel. The Fourier transform of the kernel is determined by the integral operator (2.1) acting on functions of the form $u'(s) = e^{2\pi i k s}$. Instead of computing the resulting integrals directly, we evaluate the integrals using the knowledge that the action of the operator represented by (2.1) can be computed by solving the two problems (1.11) and evaluating the jump in the normal derivatives across the x_1-axis. For data of the form $e^{2\pi i k s}$ these two problems can be solved explicitly using separation of variables. If we take the normal derivatives of the solutions we find

$$\frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{e^{2\pi i k \xi}}{(s-\xi)^2} d\xi = 4\pi \mid k \mid e^{2\pi i k s}$$

Denoting the Fourier transform by $\hat{}$ we conclude that the transform of equation (2.1) is

$$-4\pi \mid k \mid \hat{u}'(k) = \hat{R}(k) \tag{2.2}$$

The transform of the 1-D Laplacian is $-4\pi^2 k^2$ so the operator for the interface equation has the interpretation of being $-2(-\Delta_{1\text{-}D})^{\frac{1}{2}}$. Thus the motivation for preconditioners D and GM is that they are both finite difference approximations to $-2(-\Delta_{1\text{-}D})^{\frac{1}{2}}$ and hence are good approximations to the exact inverse for the flux operator for the half-space. Assuming that the operator for the half-space problem is an approximation to that for rectangles, we understand why the preconditioners work well for the case of two abutting rectangles.

BW1 and BW2 are preconditioners is defined by the operator which takes Dirichlet data on x_1-axis and maps it (via the solution of either half space problem) to Neumann data on the x_1-axis. This mapping is exactly one of the terms on the left hand side of (2.1). Thus, the transform of such a preconditioner is just

$$-2\pi \mid k \mid$$

i.e. one half the transform occurring in (2.2). Hence, for a half-plane, any discrete approximation to this mapping would be a good preconditioner. In fact, for this special problem, D and GM can be considered specific realizations of Bjorstad and Widlund's preconditioner.

In the case of two abutting rectangles, things are a little more complicated but not much. As mentioned earlier, the operator in the interface equation (1.8) is the sum of two "flux" operators, one for each domain. Consider the region Ω_1 in Figure 1. We are interested in a representation for (1.10), the operator that takes data on Γ into the normal derivative of the solution of Laplaces equation with such data as a boundary condition. As in the half-space case, one can find an explicit integral equation representation of this operator using the Green's function for a rectangle. However, by manipulating this representation (in a long and tedious calculation) one finds that the eigenfunctions of the operator are given by $\sin(\frac{k\,\pi s}{\alpha_1})$ $k = 1, 2, \cdots$. Once the eigenfunctions are known, it is a simple matter to solve Laplaces equation with such data using separation of variables. If one takes normal derivatives of these solutions and evaluates them along Γ, one finds that the eigenvalues corresponding to the operator acting on $\sin(\frac{k\,\pi s}{\alpha_1})$ are

$$-\frac{k\,\pi}{\alpha_1}\left(\frac{1+e^{\frac{-2\pi k\,\alpha_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\,\alpha_2}{\alpha_1}}}\right)$$

A similar result holds for the operator corresponding to Ω_2. From this information we can express the interface equation as

$$\left[\,\mathrm{ST}_{\alpha_1}^{-1}\,\Lambda_\alpha\,\mathrm{ST}_{\alpha_1}+\mathrm{T}_{\beta\alpha}\,\mathrm{ST}_{\beta_1}^{-1}\,\Lambda_\beta\,\mathrm{ST}_{\beta_1}\,\mathrm{E}_{\alpha\beta}\,\right]\mathrm{u}'\,(s)=\mathrm{R}(s) \tag{2.3}$$

Here ST_{α_1} and ST_{β_1} represent the sine transform of functions defined on intervals of lengths α_1 and β_1 respectively, i.e. for ST_{α_1} we extend a function defined on Γ to an odd function on a segment of twice its length and then use the sine series of the extended function. It is here we utilize the fact that we are representing our unknown function as being the difference between a linear interpolation and the solution values on the interface. This representation guarantees that u' will be continuous when extended to be odd periodic. The operator $\mathrm{E}_{\alpha\beta}$ represents extension (by zero) of a function defined on Γ to a function defined on the top side

of Ω_2. $T_{\beta\alpha}$ represents the operator that truncates a function defined on the top side of Ω_2 to a function defined on Γ. Λ_α and Λ_β are diagonal with

$$(\Lambda_\alpha)_{k,k} = -\frac{k\pi}{\alpha_1}\left(\frac{1+e^{\frac{-2\pi k\,\alpha_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\,\alpha_2}{\alpha_1}}}\right) \qquad k = 1, 2, \cdots \tag{2.4}$$

$$(\Lambda_\beta)_{k,k} = -\frac{k\pi}{\beta_1}\left(\frac{1+e^{\frac{-2\pi k\,\beta_2}{\beta_1}}}{1-e^{\frac{-2\pi k\,\beta_2}{\beta_1}}}\right) \qquad k = 1, 2, \cdots$$

We see that the eigenvalues of the flux operator corresponding to rectangles are the eigenvalues for the half plane multiplied by ratios of exponentials. In these exponential factors, the character of the domain enters in via the aspect ratio's $\frac{\alpha_2}{\alpha_1}$ and $\frac{\beta_2}{\beta_1}$. If the aspect ratios are large, then the eigenvalues are essentially those corresponding to the half plane problem. Also, we infer that effect of the geometry of the domain is most pronounced upon the eigenvalues associated with small wave numbers.

3. Model Problem

We shall now concentrate on the case when both rectangles completely share a common side ($\alpha_1 = \beta_1$ - see Figure 3.) We shall refer to this problem as the "model" problem. In this situation $E_{\alpha\beta}$ and $T_{\beta\alpha}$ and both are the identity operator. Equation (2.3) can be diagonalized; the eigenfunctions are $\sin(\frac{k\pi s}{\alpha_1})$ and the eigenvalues are given by (2.4). Furthermore, in this basis, the preconditioners described in Section 1 have a continuous analog whose transfrom is diagonal. The eigenvalues corresponding to each of the preconditioners are

$$\frac{k\pi}{\alpha_1} \qquad \text{(D, GM)} \tag{3.1}$$

$$\frac{k\pi}{\alpha_1}\left(\frac{1+e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}\right) \qquad \text{(BW1)} \tag{3.2}$$

Here $k = 1, 2, \cdots$. BW2 has a continuous analog like that of BW1, but with β_2 replacing α_2. The eigenvalues of the continuous analog of Dryja's and Golub and Mayers' preconditioners coincide since they are both finite difference approximations of the same operator.

Using (3.1), (3.2) and (2.3) we find that the eigenvalues of $M^{-1}A$ for the continuous analog of the preconditioners are

$$\left(\frac{1+e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}\right)+\left(\frac{1+e^{\frac{-2\pi k\beta_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\beta_2}{\alpha_1}}}\right) \qquad \text{(D, GM)} \tag{3.3}$$

$$1+\left(\frac{1-e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}{1+e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}\right)\left(\frac{1+e^{\frac{-2\pi k\beta_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\beta_2}{\alpha_1}}}\right) \qquad \text{(BW1)} \tag{3.4}$$

Certain properties of the preconditioners can be inferred from these expressions. For a decomposition in which the ratios $\frac{\alpha_2}{\alpha_1}$ and $\frac{\beta_2}{\beta_1}$ are sufficiently large the eigenvalues depend

very weakly on the characteristics of the domain. The eigenvalues are very close to that for the half-plane case. We expect that all three preconditioners, each of which is an approximation to the exact inverse will do very well.

The eigenvalues for D and GM are bounded from below by 2, and the maximum eigenvalue tends to infinity as the ratios $\frac{\beta_2}{\alpha_1}$ or $\frac{\alpha_2}{\alpha_1}$ tend to zero. Thus the condition number also tends to infinity as one or the other aspect ratio becomes small. We conclude that the condition number associated with preconditioners D and GM cannot be bounded independent of the domain. Similar conclusions hold for BW1 and BW2. It is worth noting that the condition numbers of BW1 and BW2 have singular behavior with respect to only one of the aspect ratios. For example, if β_2 is fixed, then for all α_2 such that $\alpha_2 \leq \beta_2$ we have $\kappa(\text{BW1}) \leq 2$. However, if α_2 is fixed, then for all $\beta_2 \leq \alpha_2$ there is no bound for $\kappa(\text{BW1})$. This implies that if we are solving a problem in which one rectangle has small aspect ratio, say Ω_1, then we should use the Neumann problem for Ω_1 as the preconditioner - BW1. The condition number of BW2 will be correspondingly large.

If we have a domain for which $(\frac{\alpha_2}{\alpha_1}) \leq (\frac{\beta_2}{\alpha_1})$ then a comparison of D (and also GM) to BW1 can easily be made. In this case, the minimum value of the eigenvalues for D is 2 and the minimum eigenvalue of BW1 is bounded below by 1. Also, every eigenvalue of D exceeds that of BW1 by

$$\left(\frac{1 + e^{\frac{-2\pi k \alpha_2}{\alpha_1}}}{1 - e^{\frac{-2\pi k \alpha_2}{\alpha_1}}} \right)$$

(This is just the ratio of (3.3) to (3.4).) From these considerations, we find the following relation for the condition numbers

$$\frac{\kappa(D)}{\kappa(BW1)} \geq \frac{1}{2}\left(\frac{1+e^{\frac{-2\pi k\,\alpha_2}{\alpha_1}}}{1-e^{\frac{-2\pi k\,\alpha_2}{\alpha_1}}}\right)$$

If $\frac{\alpha_2}{\alpha_1}$ is sufficiently small then the conditioner number associated with the continuous analogs of D will always be larger than that for BW1. If the error bound (1.5) is a reliable indicator of performance, we would expect BW1 to perform slightly better. This is intuitively correct as BW1 takes into account features of the domain, whereas and D and GM do not.

In conclusion, we see that all the preconditioners can be expected to perform rather well since each contains some approximation to the operator $-(-\Delta_{1-D})^{\frac{1}{2}}$ which is the relevant forward operator of the domain decomposition problem for the half space. The characteristics of the domain enter in weakly as long as one side is not significantly smaller than the other. If one side has a small aspect ratio, the condition number of D and GM will become large as will one of BW1 or BW2. For the model problem at least, it is wise to use the rectangle with the smaller aspect ratio to define the preconditioner suggested by Bjorstad and Widlund.

In in the discrete case (using the five point Laplacian), one can obtain a representation similar to (2.3). This shall be done in Section 5. In the case in which the two rectangles share a common side, the discrete operators can be diagonalized, and the interface equation can be solved directly - no iteration is required. One may then wonder about the usefulness of the above analysis. We shall demonstrate by numerical experiments in Section 6 that conclusions based upon a domain composed of two rectangles sharing a common side carry over more general domains. An analysis similar to that given above for the continuous operators (a comparison of preconditioners etc.) for the discrete case, has been carried out by Chan in [6].

4. A Variable Coefficient Problem

We now discuss the application of domain decomposition techniques to a variable coefficient problem. We consider a model problem which consists of solving

$$\nabla \cdot c \nabla u = f \qquad u = g \text{ on } \partial\Omega \tag{4.1}$$

on a domain Ω as in Figure 1. The coefficient c has a value c_1 in Ω_1 and a value c_2 in Ω_2. We shall dispense with a discussion of the method of domain decomposition for a discretization of (4.1) and only discuss the method from a continuous viewpoint. Again, the goal is to construct a solution from the solution of two subproblems

$$c_1 \Delta u_1 = f_1 \qquad u = g_1 \text{ on } \partial\Omega_1$$
$$c_2 \Delta u_2 = f_2 \qquad u = g_2 \text{ on } \partial\Omega_2$$

The boundary values g_1 and g_2 are not determined along Γ and an equation must be found for these undetermined values. This equation will be derived by considering conditions that the two solution u_1 and u_2 must satisfy in order to form a solution on all of Ω. First, the solution should be continuous across Γ,

$$u_1 = u_2 \text{ on } \Gamma$$

and secondly

$$c_1 \frac{\partial u_1}{\partial n} = c_2 \frac{\partial u_2}{\partial n} \tag{4.2}$$

should hold across Γ. As in the case with constant coefficients, these requirements are necessary and sufficient for u_1 and u_2 to piece together to form a weak solution. This first equation is automatically satisfied by our choice that u_1 and u_2 take the same data along Γ. It is from the second requirement that the equation for the interface values will come. Since the coefficient for each of the subproblems is constant, we can use Green's functions for each region to derive expressions for the normal derivatives of u_1 and u_2. Using such expressions in (4.2), and rearranging the resulting integrals in a way similar to the problem with constant coefficients, we arrive at an integral equation of the form

$$\int_\Gamma \left[c_1 \frac{\partial^2 G_1}{\partial n_x \, \partial n_y} + c_2 \frac{\partial^2 G_2}{\partial n_x \, \partial n_y} \right](x, s) u'(s) ds =$$

$$c_1 \frac{\partial}{\partial n_x} \left[\int_{\Omega_1} G_1(x, y) f_1(y) dy + \int_{\partial\Omega_1} \tilde{g}_1(s) \frac{\partial G_1}{\partial n_y}(x, s) ds \right]$$

$$+ c_2 \frac{\partial}{\partial n_x} \left[\int_{\Omega_2} G_2(x, y) f_2(y) dy + \int_{\partial\Omega_2} \tilde{g}_2(s) \frac{\partial G_2}{\partial n_y}(x, s) ds \right] \qquad (4.3)$$

Here we have decomposed the solution along Γ into u' and a linear function which interpolates the boundary values at the endpoints of Γ. $\tilde{g}_1$, $\tilde{g}_2$, f_1 and f_2 are defined as in the constant coefficient case. Since the geometry in this model problem is so simple, we can use Fourier analysis to express this equation in a more convenient form. The eigenfunctions of the operator on the left hand side of (4.3) are given by $\sin(\frac{k\pi s}{\alpha_1})$ $k = 1, 2, \cdots$. The operator (4.3) is diagonalizable by the sine transform, and, denoting the sine transform of a function by $\hat{}$, it takes the form

$$-\left[\frac{c_1 k \pi}{\alpha_1} \left(\frac{1 + e^{\frac{-2\pi k \alpha_2}{\alpha_1}}}{1 - e^{\frac{-2\pi k \alpha_2}{\alpha_1}}} \right) + \frac{c_2 k \pi}{\alpha_1} \left(\frac{1 + e^{\frac{-2\pi k \beta_2}{\alpha_1}}}{1 - e^{\frac{-2\pi k \beta_2}{\alpha_1}}} \right) \right] \hat{u}(k) = \hat{R}(k) \qquad (4.4)$$

In a way analogous to the constant coefficient case, $\hat{R}(k)$ is the sine transform of the right hand side of (4.3) and can be evaluated by transforming the jump across Γ of $c_i \frac{\partial \phi_i}{\partial n}$ for $i = 1, 2$. Here ϕ_i corresponds to solutions of Poisson's equations in Ω_1 and Ω_2. Using this formulation we shall examine a preconditioner which is an extension of the ideas of Bjorstad and Widlund and is suggested by Bramble, Pasciak and Hubbard in [3]. As in the constant coefficient coefficient case, each of the operators occurring on the left hand side of (4.3) has the interpretation as being a map from Dirichlet data on Γ to Neumann data on Γ. The idea is to use one of the operators as a preconditioner. The inverse of the preconditioner (the operator of most interest to us) can be implemented by just solving a Neumann problem on one of the domains.

For our model problem, the eigenvalues of such a preconditioner are given by one or the other of the terms on the right hand side of (4.4). Let M_1 and M_2 denote preconditioners defined in terms of the Neumann problem corresponding to Ω_1 and Ω_2 respectively. The eigenvalues corresponding to the matrix $M_i^{-1}A$ are

$$1 + \frac{c_2}{c_1}\left(\frac{1 - e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}{1 + e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}\right)\left(\frac{1 + e^{\frac{-2\pi k\beta_2}{\alpha_1}}}{1 - e^{\frac{-2\pi k\beta_2}{\alpha_1}}}\right) \qquad (M_1)$$

$$1 + \frac{c_1}{c_2}\left(\frac{1 + e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}{1 - e^{\frac{-2\pi k\alpha_2}{\alpha_1}}}\right)\left(\frac{1 - e^{\frac{-2\pi k\beta_2}{\alpha_1}}}{1 + e^{\frac{-2\pi k\beta_2}{\alpha_1}}}\right) \qquad (M_2)$$

As in the constant coefficient case we see that the geometry of the domain has a weak effect as long as the aspect ratios are large. For fixed aspect ratios we can clearly see the effect of the coefficients on the condition number. In particular, if $c_2 \ll c_1$ then the condition number associated with M_1 will be very close to 1. (Similarly, if $c_2 \ll c_1$ then $\kappa(M_2) \equiv 1$.) This suggests that one should use the domain associated with the larger coefficient to define the preconditioner. If one uses the "wrong" side to define the preconditioner, then we expect a larger condition number. (This will of course depend on the aspect ratios as well.)

The ability to completely diagonalize the system of equations (4.3) using Fourier analysis carries over into the discrete case if one uses the standard five point difference approximation and a uniform grid. Thus, the discrete equations for the model problem considered here can be solved directly. One could consider the solution of this model problem as a preconditioner for the case in which the region is like that in Figure 1. (Just extend Ω_1 into Ω_2.) We denote this preconditioner by EV. (For extension - variable.) Some numerical experiments using M1, M2, and EV will be presented in the next Section.

For a general variable coefficient elliptic problem, Bramble, Pasciak and Hubbard suggest using a preconditioned conjugate gradient algorithm in which the preconditioner is

defined in terms of a variable coefficient "approximate" equation [3]. The approximate equation is one chosen so that the techniques of domain decomposition can be readily applied. One possible approximate problem consists of decomposing the domain into subregions and defining the coefficients of the approximate operator to be constant on each of the sub regions. For solving this approximate equation the preconditioner EV may be useful.

5. Derivation of the Discrete Operators

We now derive the discrete eigenvalues and eigenvectors corresponding to the capacitance matrix C in (1.4). For alternative derivations see [1], [6] or [7]. As discussed in Section 2, the continuous analog of this matrix is an operator composed of two parts. Each part is associated with a domain and represents the mapping, defined in terms of a solution of Laplaces equation, of Dirichlet data on Γ to Neumann data (or flux) on Γ. It is the same in the discrete case, and therefore it suffices to find the eigenvalues and eigenvectors of the discrete operator for one rectangle. Our derivation will essentially follow the procedure for the continuous case given in Section 2. Consider the rectangle in Figure 4. Given data on Γ, the flux induced at a point $(lh_x, 0)$ is given by

$$\frac{1}{2h_x}\left[\phi_{l-1,0} - 2\phi_{l,0} + \phi_{l+1,0}\right] + \frac{1}{h_y}\left[\phi_{l,1} - \phi_{l,0}\right] \tag{5.1}$$

applied to, ϕ, a solution of

$$\Delta^h \phi = 0 \quad \text{on } \Omega_1 \tag{5.2}$$

$$\phi = g \quad \text{on } \Gamma$$

Here we use Δ^h to denote the standard five point Laplacian. h_x and h_y denote the mesh widths in the x and y directions respectively.

We recognize the second term of (5.1) as being the finite difference analog of the normal derivative. The first term is due to the flux induced by values along Γ and does not appear in the continuous case. However, the inclusion of it is natural if one considers the derivation of the five-point difference stencil from a control volume point of view.

The eigenvalues for the operator represented by this mapping are $\sin(\frac{\pi k l h_x}{\alpha_1})$ i.e. the continuous case eigenvalues evaluated at the nodes. This can be deduced from a manipulation of the discrete Green's function for the rectangle, or from matrix considerations ([1],[6],[7]).

We now precede to find the eigenvalues associated with such eigenfunctions. We shall do this by solving (5.2) with eigenfunction data, and then apply the difference operator (5.1)

to the result. Our solution of (5.2) will be constructed from solutions for a domain which has $\alpha_2 = \infty$. These solutions will be combined in such a way that the boundary condition along the top of the rectangle will be satisfied.

For the half-infinite rectangle, we employ a discrete version of separation of variables and seek solutions of the form

$$\lambda_k^m \, e^{\frac{2\pi i l k h_x}{2\alpha_1}} \tag{5.3}$$

We have denoted the dependence of λ on the wavenumber k by adding the subscript. The results for $\sin\left(\frac{\pi k l h_x}{\alpha_1}\right)$ will be obtained by combining solutions of the form (5.5) for k and $-k$. We determine λ_k by substituting (5.3) into the difference equation (5.2). At the point $(lh_x\,,\, mh_y)$ we find

$$e^{\frac{2\pi i l k h_x}{2\alpha_1}} \lambda_k^m \frac{\left(e^{\frac{2\pi i k h_x}{2\alpha_1}} - 2 + e^{\frac{-2\pi i k h_x}{2\alpha_1}}\right)}{{h_x}^2} + e^{\frac{2\pi i l k h_x}{2\alpha_1}} \lambda_k^m \frac{\left(\lambda_k^{-1} - 2 + \lambda_k\right)}{{h_y}^2} = 0$$

If we cancel the terms $e^{\frac{2\pi i l k h_x}{2\alpha_1}}$ and λ_k^m and simplify, we arrive at an equation for λ_k

$$\lambda_k^2 - \left(2 + 4\left(\frac{h_y}{h_x}\right)^2 \sin^2\left(\frac{\pi k h_x}{2\alpha_1}\right)\right)\lambda_k + 1 = 0 \tag{5.4}$$

We use λ_k corresponding to the root of (5.4) which is less than 1 (to get a solution bounded at ∞). Thus,

$$\lambda_k = \frac{\left[2+4\left(\frac{h_y}{h_x}\right)^2 \sin^2\left(\frac{\pi k h_x}{2\alpha_1}\right)\right] - \sqrt{\left[2+4\left(\frac{h_y}{h_x}\right)^2 \sin^2\left(\frac{\pi k h_x}{2\alpha_1}\right)\right]^2 - 4}}{2}$$

The solution for the finite rectangle is now an infinite sum of solutions of the form (5.3) with λ_k defined above,

$$w_{l,m} = e^{\frac{2\pi i l k h_z}{2\alpha_1}} [\lambda_k^m + \sum_{p=2,4,6,\cdots} (\lambda_k^{Np+m} - \lambda_k^{Np-m})] \tag{5.5}$$

$$= e^{\frac{2\pi i l k h_z}{2\alpha_1}} [\lambda_k^m + (\lambda_k^m - \lambda_k^{-m})\left(\frac{1}{1-\lambda_k^{2N}} - 1\right)]$$

Here N is the number of panels along the side, i.e. $\frac{\alpha_2}{h_y}$. Formula (5.5) is derived using the method of images. The simplification is obtained by using the summation formula for a geometric series.

The difference operator (5.1) is now applied to the solution (5.5) along Γ and this yields

$$C\, e^{\frac{2\pi i l k h_z}{2\alpha_1}} = \Lambda_k\, e^{\frac{2\pi i l k h_z}{2\alpha_1}} \tag{5.6}$$

with

$$\Lambda_k = \left[\frac{-2}{h_z}\sin^2(\frac{\pi k h_z}{2\alpha_1}) + \frac{1}{h_y}\left\{(\lambda_k - \lambda_k^{-1})\left[\frac{1}{1-\lambda_k^{2N}} - 1\right] + (\lambda_k - 1)\right\}\right]$$

The eigenvalue Λ_k is also an eigenvalue for data of the form $e^{\frac{-2\pi i l k h_z}{2\alpha}}$ and therefore will be an eigenvalue for a linear combination of the two, i.e. $\sin(\frac{\pi k l h_z}{\alpha_1})$. Thus Λ_k with $k = 1, 2, \cdots$ are the eigenvalues that we seek.

Using these eigenvalues and eigenvectors it is a simple matter to compute the discrete operator corresponding to (1.4). If the domain is that given in Figure 1, then we use the formula (2.3) with discrete sine transforms replacing the continuouc sine transforms. We have a representation of the matrix equations as

$$\left[\, A_{33} - A_{13}^t A_{11}^{-1} A_{13} - A_{23}^t A_{22}^{-1} A_{23} \,\right] X_3 = \tag{5.7}$$

$$\left[\, ST_{\alpha_1}^{-1}\, \Lambda_\alpha\, ST_{\alpha_1} + T_{\beta\alpha}\, ST_{\beta_1}^{-1}\, \Lambda_\beta\, ST_{\beta_1}\, E_{\alpha\beta} \,\right] X_3 = F_3 - A_{13}^t A_{11}^{-1} F_1 - A_{23}^t A_{22}^{-1} F_2$$

The operators and vectors on the right hand side of (5.7) are those defined in (1.4).

ST_{α_1}, ST_{β_1} represent the discrete sine transform of functions defined on intervals of lengths α_1 and β_1 respectively. The operator $E_{\alpha\beta}$ represents extension (by zero) of a function defined on Γ to a function defined on the top side of Ω_2. $T_{\beta\alpha}$ represents the operator that truncates a function defined on the top side of Ω_2 to a function defined on Γ. Λ_α is diagonal with its (k,k)th entry given by Λ_k in (5.6). Λ_β is diagonal and has entries similar to those of Λ_α but with β_1 used instead of α_1. The right hand side of the equation can be formed by evaluating the jump in flux induced by the solution of two Poisson equations. (See the discussion concerning equations (1.8) and (1.9).) In the iterative solution of equations (1.4) it is only necessary to compute the product of C with a vector. From (5.7) we see that this can be done completely using discrete sine transforms. Also, if $\beta_1 = \alpha_1$ then the discrete operator can be diagonalized and the equation (1.4) can be solved directly.

For the discrete half space case, the eigenvalues are the same as that for the rectangle except that $N = \infty$. If $h_x = h_y = h$, then the eigenvalues Λ_k simplify and are given by

$$\Lambda_k = \frac{1}{h}\sqrt{4\sin^2(\frac{\pi kh}{2\alpha_1}) + 4\sin^4(\frac{\pi kh}{2\alpha_1})} \tag{5.8}$$

$$= \frac{1}{2}\sqrt{4K + h^2K^2}$$

Since the eigenvalues for the full half space operator are twice those of (5.8), we see that Golub and Mayers' preconditioner is the exact inverse for the discrete half space problem. For more comparisons of the discrete preconditioners, see Chan [6].

6. Numerical Experiments

We now discuss the results of some numerical experiments. In all cases the discretization will be that arising from a five-point difference approximation to the Laplacian. The mesh is uniform with width h in each direction. Since we are using a regular distribution of points and the interface is a straight line, the iterative solution of equation (1.4) can be carried out using fast sine transforms.

In the first experiment we worked with a domain as in in Figure 1. We were interested in the behavior of the condition number of $M^{-1}A$ for each of the preconditioners as the ratio $\frac{\alpha_1}{\beta_1}$ was varied, i.e. as the region becomes more "L" shaped. The goal is to see if the conclusions based on the model problem were valid for perturbed domains. We examined five different pre-conditioners; D,GM, BW1, BW2, and one which was the exact solution for the domain consisting of the extension of the narrower rectangle into the larger rectangle. We shall refer to this as preconditioner EC (Extension-Constant). Chan in [6] has also recommended this as a preconditioner. Using the results of the previous Section, the application of all of the preconditioners can be completely carried out using fast sine transforms. When $\frac{\alpha_1}{\beta_1} = 1$ then we are solving the model problem. The results of the experiment are plotted in Figure 5. From these results we see that, except for BW2, the condition number given by the model problem accurately predicts the value of the condition number over a wide range of the ratio $\frac{\alpha_1}{\beta_1}$. It appears that the condition number appears bounded with respect to perturbations in this ratio. Except for minor deviations, BW1 and G have about the same condition number while that associated with D is slightly larger. BW2 appears to be worse than all the other preconditioners. The difference between κ(D) and κ(G) perhaps lies in the accuracy with which each approximates the discrete half-space flux operator. In this experiment we chose the component rectangles such that $\alpha_2 = \beta_2$. Some experiments with $\alpha_2 \neq \beta_2$ were performed, and it was found that the results did not differ significantly. One of these, that in

which $\frac{\alpha_2}{\beta_2} = .1$ is presented in Figure 6.

In the second experiment we again used a domain like that in Figure 1, but varied the the ratio $\frac{\alpha_2}{\alpha_1}$. As discussed in the Section 3, we expect that as this ratio tends to zero the condition number associated with BW2, D and GM should grow. The results for two different width fingers is presented in Figures 7 and 8. For a finger which is 3/5 the width of the bottom rectangle, we indeed see the expected growth of the condition number as the aspect ratio $\frac{\alpha_2}{\alpha_1}$ was made smaller. κ(BW1) grows, but does not appear to do it catastrophically as it does with D, GM and BW2. We also note that our fifth preconditioner EC, does very well - in fact, the condition number gets smaller. For a very thin finger, one which is 1/5 the width of the bottom rectangle, the growth of the condition number occurs, but it is not as marked as the previous case. For very thin fingers, the preconditioners appear to become uniformly better.

The above estimates of the condition number were primarily made to assess the predictive power of our analysis for the model problem. However, the condition number is only used in an error bound, and it may not necessarily indicate the true performance of the iterative method. To see if the condition number is indeed a reliable indicator, we determined the number of conjugate gradient iterations that were required to reduce the increment to a value less than $C' \ h^2$. (By increment we mean the amount added to the current approximation to obtain the next one.) This stopping criterion was chosen since we were interested in solving the equations to an accuracy which was a multiple of the truncation error of the numerical difference scheme. We choose the constant sC' (somewhat arbitrarily) to be 10^{-1}. The domain chosen was that which gave rise to Figure 7. These results are presented in Table 1. As the condition number of every preconditioner is constant for a wide range of the parameter it is no surprise that the number of iterations was essentially constant as well. In the cases where the condition number was large, there was an increase in the number of

iterations. (One or two more iterations.) It appears that EC does uniformly well over the complete range of the aspect ratio. The results for GM and BW1 are almost identical to EC except for fairly small aspect ratios. D and BW2 are a bit worse, requiring on the order of two more iterations than EC. Compared to the results using no preconditioning, all the preconditioners do rather well.

In the last experiment we investigated the effect of variable coefficients. The domain was a region as in Figure 1. The width of the centered finger was 3/5 that of the bottom. (The results for non-centered fingers were nearly identical.) Three preconditioners were used. The first two were M1 and M2 of Section 4; preconditioners defined in terms of Neumann problems on Ω_1 and Ω_2 respectively. The third one, EV of Section 4, was the exact solution operator for a domain which consisted of Ω_1 extended into Ω_2 and using the coefficients of Ω_2 for the operator in the extension. Due to the simple geometry, each of these preconditioners and the iterative scheme for solving the capacitance matrix equations was implemented using fast sine transforms. Our stopping criterion for the iterative scheme was the same as that for the constant coefficient case. In the first experiment $c_1 = 100.0$ and $c_2 = 1.0$. In Table 2 we present the number of iterations verses the ratio $\frac{\alpha_2}{\alpha_1}$ for each preconditioner and the standard (unconditioned) conjugate gradients. As discussed in Section 4, the coefficients for this problem can be expected to reduce the condition number of M1. This indeed appears to be the case, as the number of iterations it takes, 3, is only one more than can be expected with the exact inverse as a preconditioner. Our suggested preconditioner EV did as well, while using M2 was particularly bad. The results for M2 are expected since the aspect ratio as well as the coefficients have an unfavorable effect on the condition number. In the second experiment, $c_1 = 1.0$ and $c_2 = 100.0$. The results are presented in Table 3. The effect of the coefficients can be seen by an increase in the number of iterations. For M1, depending on the aspect ratio, it either doubled or tripled the number of iterations. This clearly suggests that if one uses a preconditioner defined in terms of a Neumann problem, the Neumann problem should

be the one associated with the region with a larger coefficient. The preconditioner EV was also effected by these values of the coefficients, however, the effect was slight. The number of iterations required for convergence was only one more than in the first experiment with variable coefficients. For M2, the coefficients in the second experiment were favorable, and we note a decrease in the number of iterations. In every case the use of preconditioners dramatically reduced the number of iterations compared to unconditioned conjugate gradients. As for the for a comparison of the preconditioners, we find that EV does as well as the best of M1 or M2. EV appears to work well independently of the configuration of the coefficients while M1 and M2 do not.

7. Conclusions

We have presented another view of domain decomposition, one which may be illuminating to those who find the discrete presentation of the method difficult to understand. Using this view, we have analyzed several preconditioners associated with the method. We conclude, at least for the model problem, that the preconditioners are essentially the same. Each of them contains some approximation to $-(-\Delta_{1-D})^{\frac{1}{2}}$, which is the principle part of the forward operator of the interface equations. If the aspect ratios of the component rectangles are not excessively small, then each of the preconditioners can be expected to do rather well.

We have also examined a particular variable coefficient problem and introduced a new preconditioner suitable for it. One key fact which can be inferred from our analysis is that piecewise constant coefficient problems are readily amenable to solution using domain decomposition ideas. Although we examined a very simple problem, the basic ideas carry over to more complicated problems. In particular, insight obtained from the model problem may be useful for those implementing the algorithm of Bramble Pasciak and Hubbard in [2] or [3].

Our numerical experiments demonstrated that the model problem is an accurate predictor of the behavior of the method "L" shaped and approximately "L" shaped domains. In the case of variable coefficients, we also confirmed that conclusions based on the model problem were correct.

Bibliography

[1] P. Bjorstad and O. Widlund, "Iterative Methods for the Solution of Elliptic Problems on Regions Partitioned into Substructures", Technical Report #136, New York University, New York, 1984.

[2] J. Bramble, J. Pasciak, and A. Schatz, "The construction of Preconditioners for Elliptic Problems by Substructuring I", manuscript, 1985.

[3] J. Bramble, J. Pasciak, and A. Schatz, "An iterative Method for Elliptic Problems on Regions Partitioned into Substructures", manuscript, 1985.

[4] P. Concus, G.H. Golub, and D.P. O'Leary, "A generalized conjugate gradient method for the numerical solution of elliptic partial differential equations", *Sparse Matrix Computations*, J.R. Bunch and D.J Rose, eds., Academic Press, New York, 1976, pp. 309-332.

[5] P. Concus, G.H. Golub, and G. Meurant, "Block Preconditioning for the Conjugate Gradient Method", SIAM J. Sci. Stat. Comput., vol. 6, #1, (1985), pp. 220-252.

[6] T. Chan, "Analysis of Preconditioners for Domain Decomposition" Research Report YALEU/DCS/RR-408, Dept. of Computer Science, Yale University, 1985.

[7] T. Chan and D. Resasco, "A Domain Decomposed Fast Poisson Solver on a Rectangle", Technical Report YALEU/DC/TR-409, Dept. of Computer Science, Yale Univ., 1985.

[8] M. Dryja, "A Capacitance Method for Elliptic Problems on Regions Partitioned into Subregions", Numer. Math., 32 (1982), pp.51-64.

[9] G.H Golub and D. Mayers, "The use of Pre-Conditioning over Irregular Regions" Lecture at Sixth Int. Conf. on Computing Methods in Applied Sciences and Engineering, Versailles, Dec. 1983.

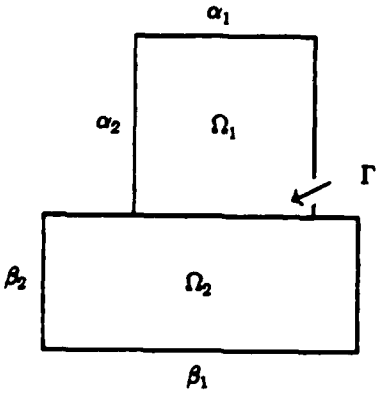


FIGURE 1

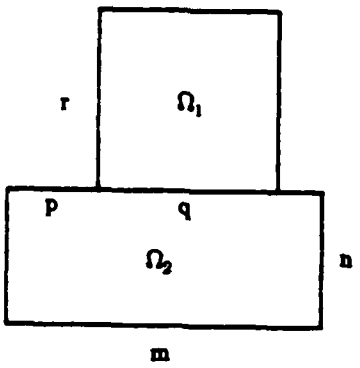


FIGURE 2

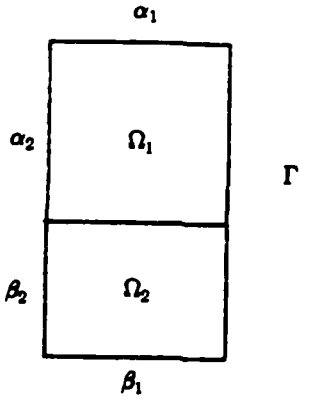


FIGURE 3

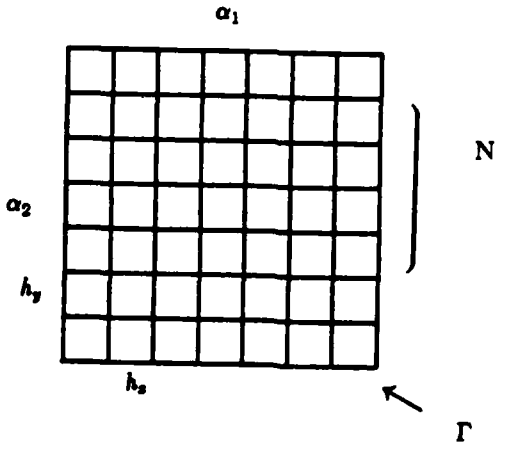


FIGURE 4

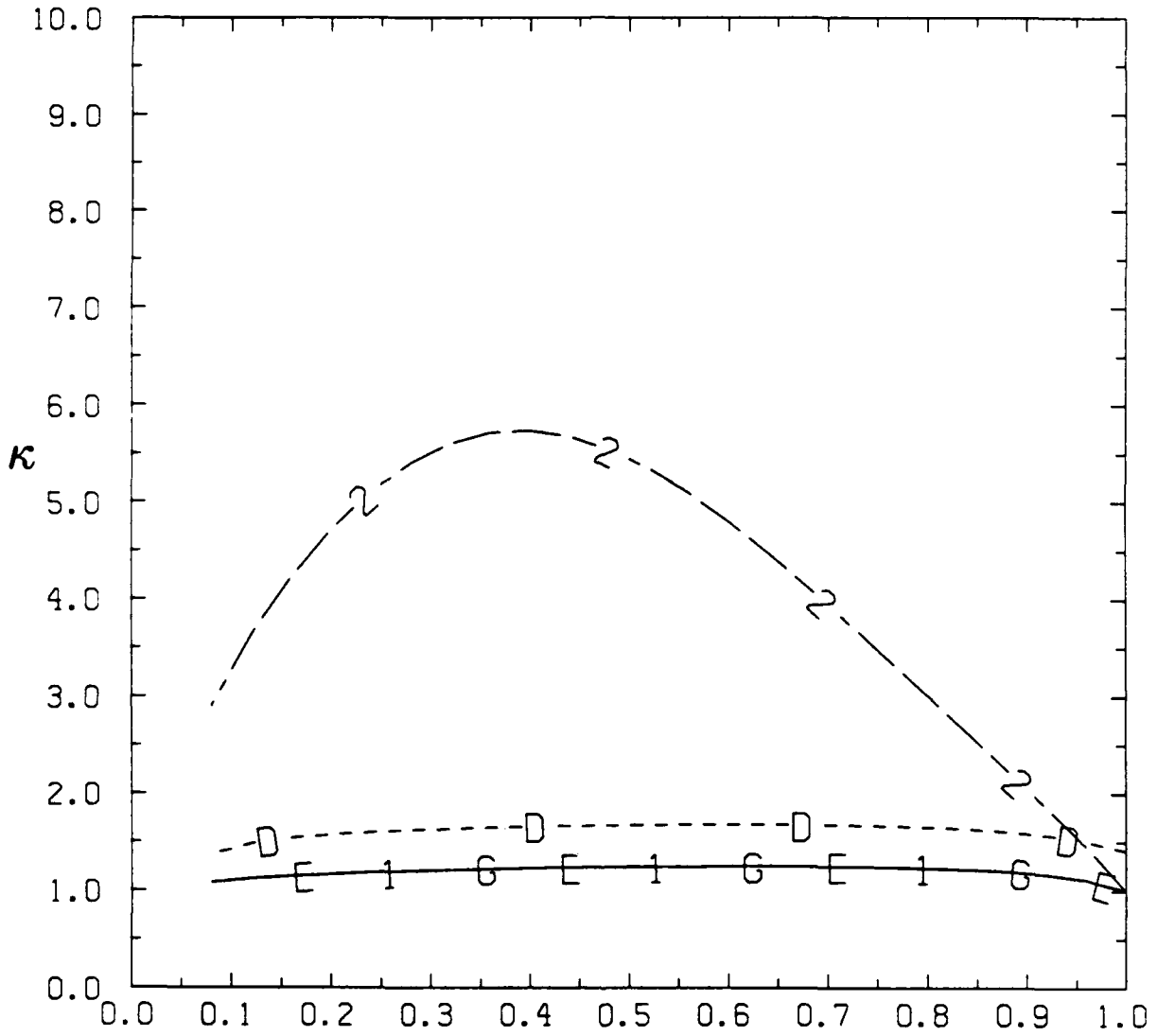


Figure 5

Condition number vs. $\frac{\alpha_1}{\beta_1}$.

$\beta_1 = 1.0 \quad \alpha_2 = 1.0 \quad \beta_2 = 1.0$

$h = .02$

E = EC D = D G = GM

1 = BW1 2 = BW2

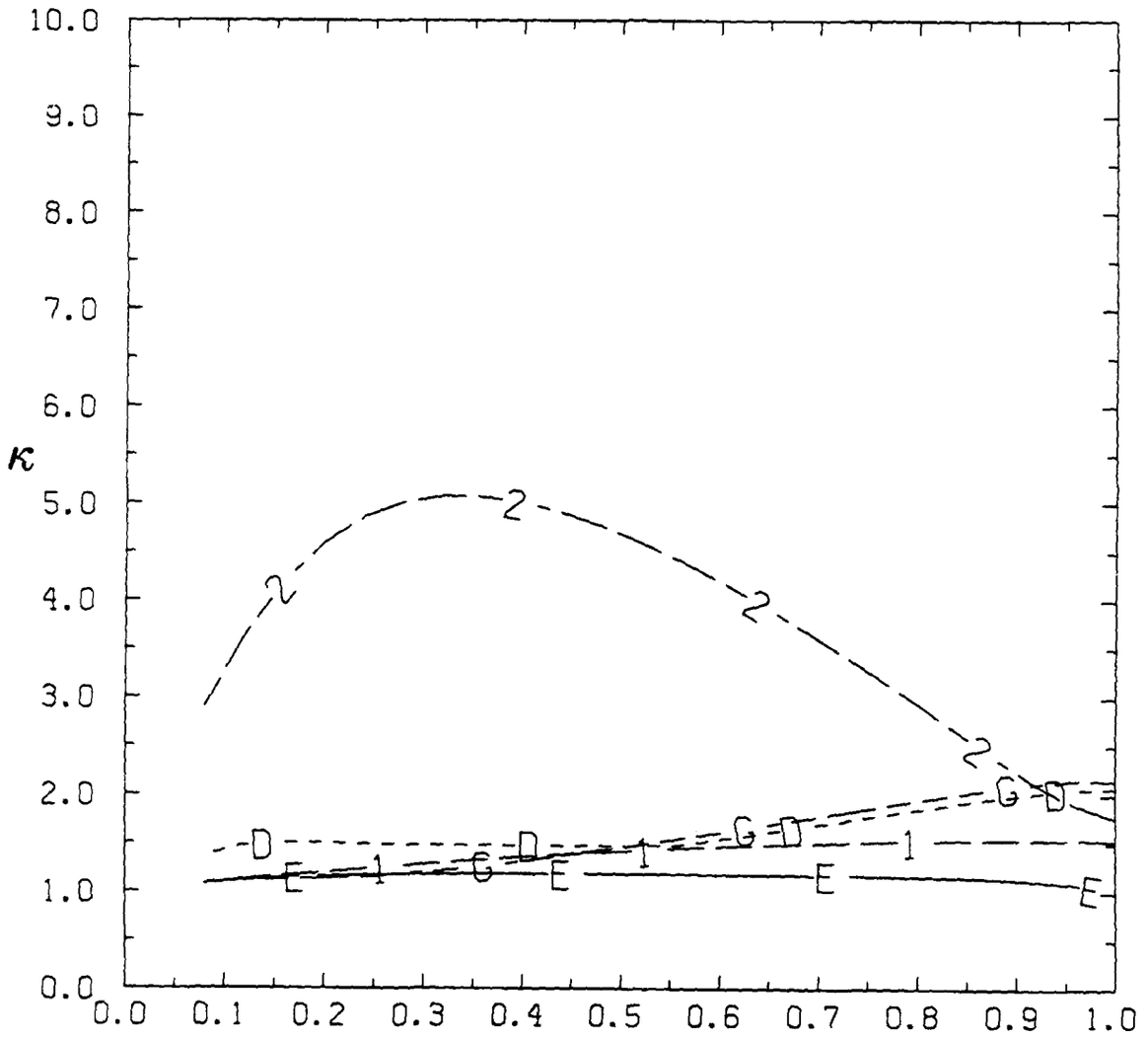


Figure 6

Condition number vs. $\frac{\alpha_1}{\beta_1}$.

$\beta_1 = 0.1 \quad \alpha_2 = 1.0 \quad \beta_2 = 1.0$

$h = .02$

E = EC D = D G = GM

1 = BW1 2 = BW2

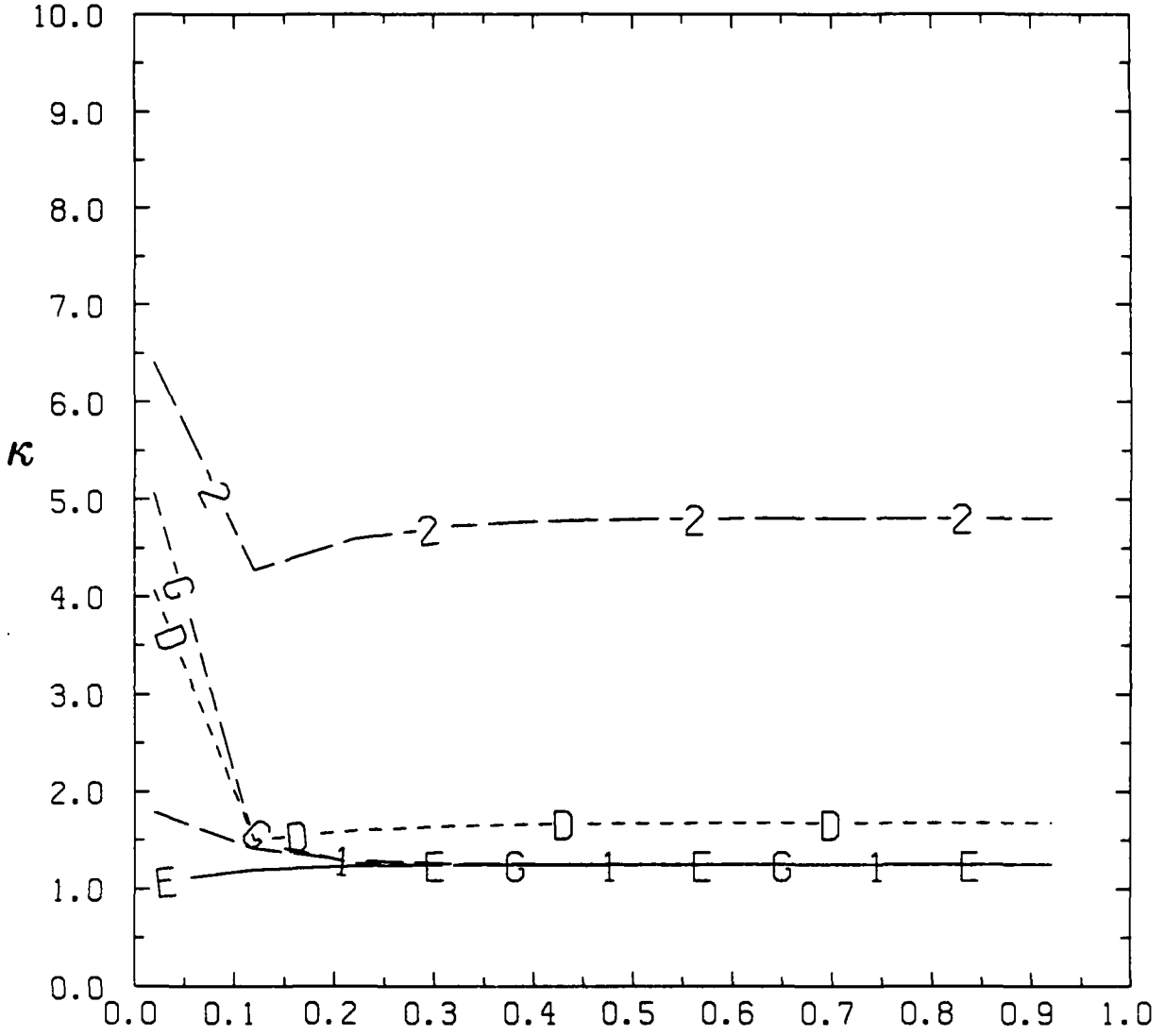


Figure 7

Condition number vs. $\frac{\alpha_2}{\alpha_1}$.

$\alpha_1 = 0.6$ $\beta_2 = 1.0$ $\beta_1 = 1.0$

Finger centered on top rectangle.

$h = .02$

E = EC D = D G = GM

1 = BW1 2 = BW2

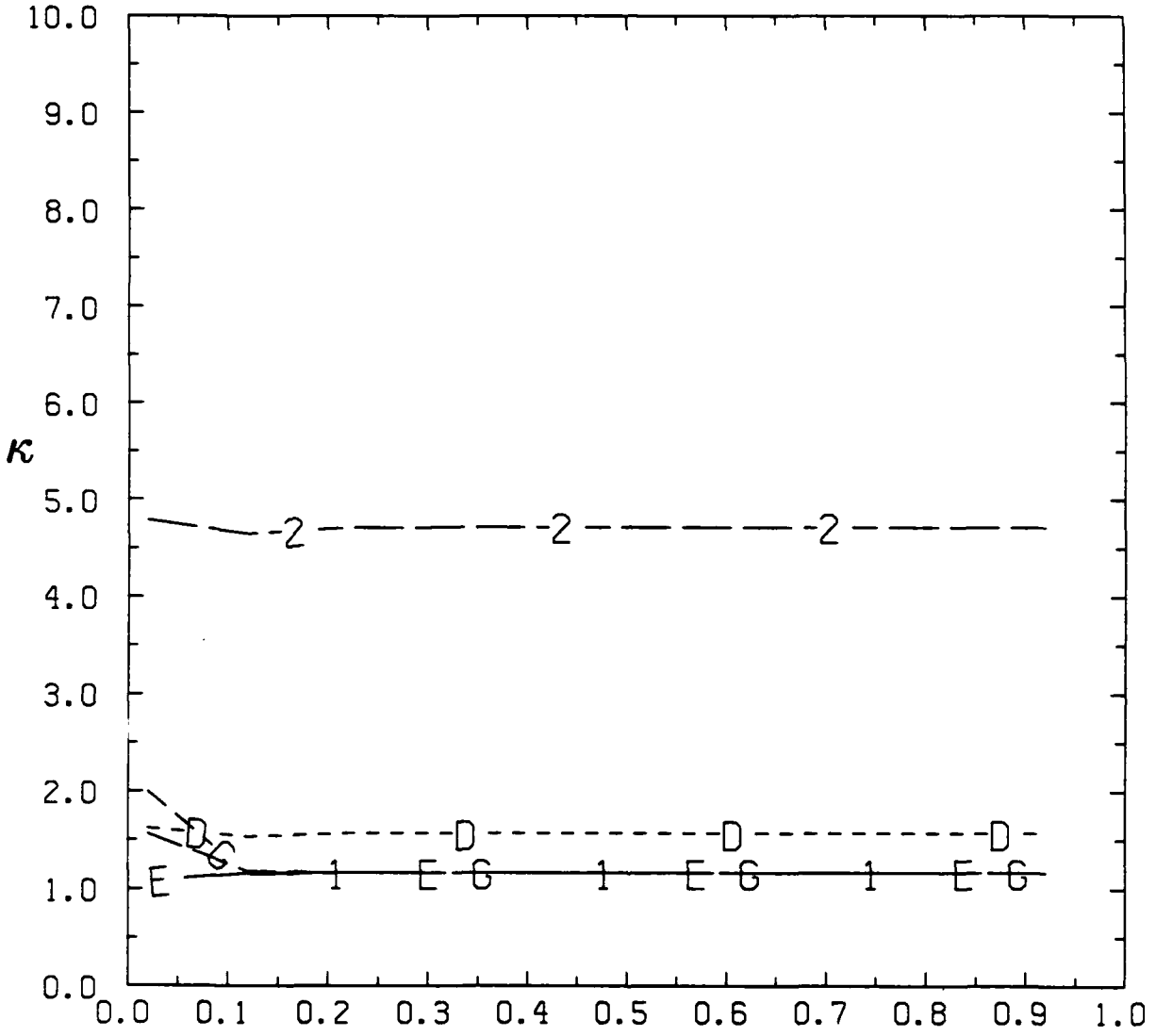


Figure 8

Condition number vs. $\frac{\alpha_2}{\alpha_1}$.

$\alpha_1 = 0.2$ $\beta_2 = 1.0$ $\beta_1 = 1.0$

Finger centered on top rectangle.

$h = .02$

E = EC D = D G = GM

1 = BW1 2 = BW2

Ratio	Preconditioner					
$\frac{\alpha_2}{\alpha_1}$	EC	BW1	BW2	D	GM	Identity
0.05000	4	8	5	5	6	28
0.10000	4	6	5	5	5	29
0.15000	4	6	5	5	4	31
0.20000	4	6	4	5	4	31
0.25000	4	6	4	5	4	31
0.30000	4	6	4	6	4	32
0.35000	4	6	4	6	4	32
0.40000	4	6	4	6	4	32
0.45000	4	6	4	6	4	32
0.50000	4	6	4	6	4	30

Table 1

Number of iterations vs ratio $\frac{\alpha_2}{\alpha_1}$

$\alpha_2 = .6 \quad \beta_1 = 1.0 \quad \beta_2 = 1.0$

$h = .01$

Constant coefficients

Ratio	Preconditioner			
$\frac{\alpha_2}{\alpha_1}$	EV	M2	M1	Identity
0.05000	3	10	3	29
0.10000	3	8	3	33
0.15000	3	8	2	34
0.20000	3	8	3	35
0.25000	3	7	3	35
0.30000	3	7	3	35
0.35000	3	7	3	39
0.40000	3	7	3	35
0.45000	3	7	3	39
0.50000	3	7	3	35

Table 2

Number of iterations vs ratio $\frac{\alpha_2}{\alpha_1}$

$\alpha_2 = .6 \quad \beta_1 = 1.0 \quad \beta_2 = 1.0$

$h = .01$

Variable coefficients $c_1 = 100.0 \quad c_2 = 1.0$

Ratio	Preconditioner			
$\frac{\alpha_2}{\alpha_1}$	EV	M2	M1	Identity
0.05000	5	5	9	38
0.10000	5	5	7	37
0.15000	5	5	6	42
0.20000	5	5	6	42
0.25000	5	5	6	42
0.30000	5	5	6	38
0.35000	5	5	6	42
0.40000	5	5	6	42
0.45000	5	5	6	37
0.50000	5	5	6	38

Table 3

Condition number vs. $\frac{\alpha_2}{\alpha_1}$.

$\alpha_2 = .6 \quad \beta_1 = 1.0 \quad \beta_2 = 1.0$

$h = .01$

Variable coefficients. $c_1 = 1.0 \quad c_2 = 100.0$

END

FILMED

2-86

DTIC